μ-Synthesis Design of A Half-Car Active Suspension System

Robust Control – Final Project

Course: Robust Control (Spring 2021)

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Contents

1. Introduction

(1) Course

Robust Control (Spring 2021)

(2) Objective

To obtain robust performance, in order to minimize the sprung mass (chassis) acceleration and to ensure road-holding characteristics.

2. μ Synthesis

(1) Structured Singular Value

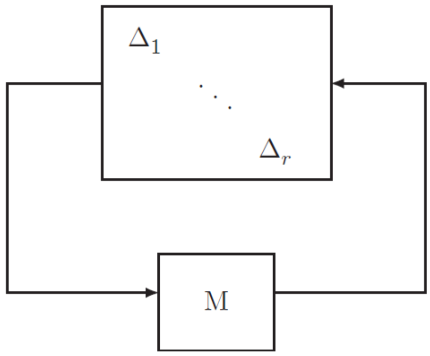


Figure 1: System with structured uncertainty [4]

Definition 17.1 *For any given matrix $M \in \mathbb{C}^{n\times n}$, the structured singular value $\mu_{\mathbf{\Delta}}(M)$ is defined as*

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\sigma_{\max}(\Delta) \mid \Delta \in \mathbf{\Delta},\ \det(I - M\Delta) = 0\}}. \tag{17.10}$$

$\mu_{\mathbf{\Delta}}(M) = 0$ *when there is no* $\Delta \in \mathbf{\Delta}$ *satisfying* $\det(I - M\Delta) = 0$.

17.3.1.1 Single Scalar Block Uncertainty $\Delta = \{\delta I \,|\, \delta \in \mathbb{C}\}$

In this case, $\mu_{\Delta}(M) = \rho(M)$ holds. Here, $\rho(M)$ denotes the spectral radius of matrix M, that is, the maximum absolute value of all eigenvalues of M.

Proof. First, $\det(I - M\delta) = \det(\delta^{-1}I - M)\det(\delta I) = 0$ holds. So any nonzero δ^{-1} satisfying this equation is an eigenvalue of M. Then, it is easy to see that the reciprocal of the minimum size of uncertainty δ satisfying this equation is the spectral radius $\rho(M)$ of matrix M, that is,

$$\mu_{\Delta}(M) = \frac{1}{\min(|\delta| : \det(I - M\delta) = 0)}$$

(2) Robust Stability and Robust Performance

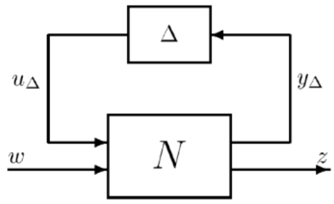


Figure 2: NΔ-structure for robust performance analysis [4]

$$\begin{aligned}
\text{NS} &\Leftrightarrow N \text{ (internally) stable} \\
\text{NP} &\Leftrightarrow \bar{\sigma}(N_{22}) = \mu_{\Delta_P} < 1, \ \forall\omega, \text{ and NS} \\
\text{RS} &\Leftrightarrow \mu_{\Delta}(N_{11}) < 1, \ \forall\omega, \text{ and NS} \\
\text{RP} &\Leftrightarrow \mu_{\tilde{\Delta}}(N) < 1, \ \forall\omega, \ \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}, \text{ and NS}
\end{aligned}$$

Where NS means nominal stability; NP means nominal performance; RS means robust stability; RP means robust performance.

(3) μ synthesis and DK-iteration

$\max \rho(QN) \leqq \mu(N) \ \leqq \inf \bar{\sigma}(DND^{-1})$

The above relationship shows μ's upper and lower bound. However, only the upper bound is a convex problem. Therefore, we only need to find $\min_{K}(\min_{D\in\mathcal{D}} \|DN(K)D^{-1}\|_{\infty})$.

(a) DK-iteration

1. **K-step.** Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem, $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed $D(s)$.
2. **D-step.** Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

3. Problem Statement

Figure 3: A Car

Cars are part of our lives. However, sometimes we don't experience a comfortable ride. Why? Does the problem result from passengers? Or is the problem caused by the suspension system?

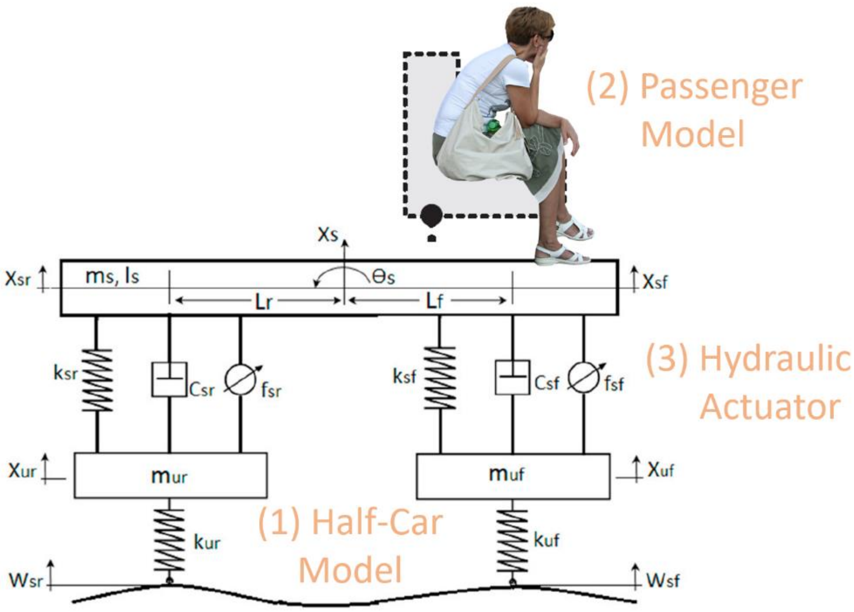
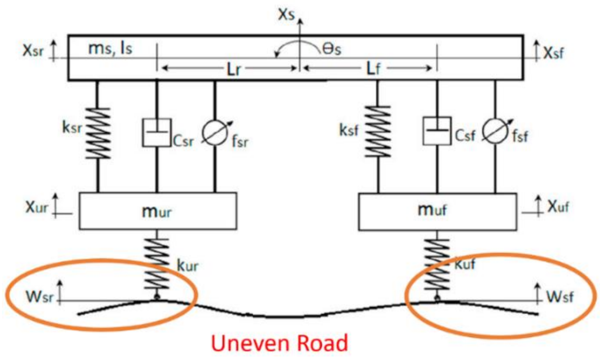


Figure 4: A Half-Car Active Suspension System

(1) Problem

(a) Uneven Road



(b) Uncertainties in System Models

- Uncertainties exist in passenger models and suspension systems.

(2) Ride Comfort

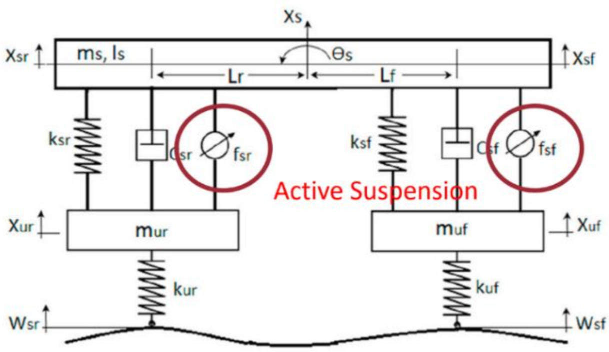
- $\ddot{X}_s$ represents ride comfort of a vehicle.

(3) Suspension Deflection

- $(X_{sf} - X_{uf})$ represents the suspension deflection.
- $(X_{sr} - X_{ur})$ represents the suspension deflection.

(4) Purpose of This Project

- To design a stabilizing controller to control the actuators
- To improve both ride comfort and suspension deflections.

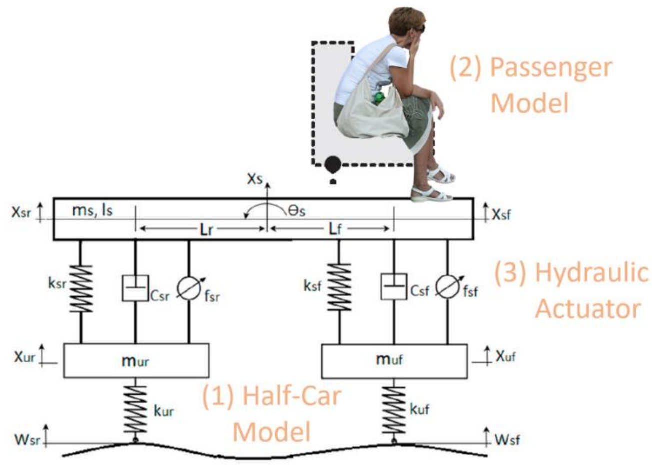


(5) Performance Specification

- The structured singular value $\mu < 1$.

4. System Structure

A Half-Car Active Suspension System
= Half-Car Model + Passenger Model + Hydraulic Actuator



These three sub-systems can be found in the paper [1], [2], and [3] for the half-car model, passenger model, and hydraulic actuators, respectively.

(1) Half-Car Model

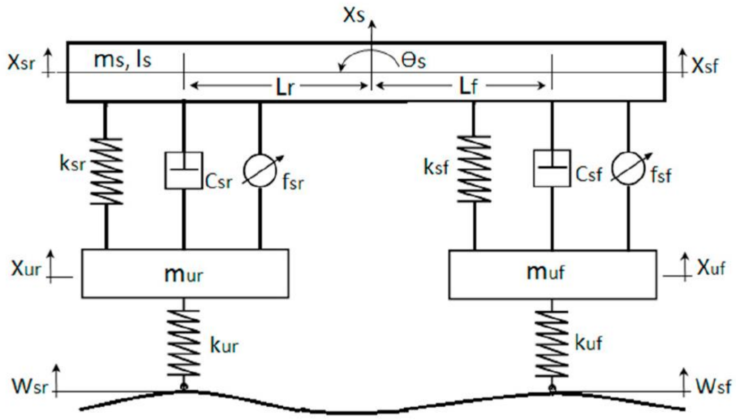


Figure 5: Half Car 4-DOF Active Suspension Model

(a) Equations of Motion

$$m_s \ddot{X}_s + c_{sf}(\dot{X}_{sf} - \dot{X}_{uf}) + k_{sf}(X_{sf} - X_{uf}) + c_{sr}(\dot{X}_{sr} - \dot{X}_{ur}) + k_{sr}(X_{sr} - X_{ur}) - f_{sf} - f_{sr} = 0 \quad (1)$$

$$I_s \ddot{\Theta}_s + L_f \left[c_{sf}(\dot{X}_{sf} - \dot{X}_{uf}) + k_{sf}(X_{sf} - X_{uf}) - f_{sf}\right] - L_r \left[c_{sr}(\dot{X}_{sr} - \dot{X}_{ur}) + k_{sr}(X_{sr} - X_{ur}) - f_{sr}\right] = 0 \quad (2)$$

$$m_{uf} X_{uf} - c_{sf}(X_{sf} - X_{uf}) - k_{sf}(X_{sf} - X_{uf}) + k_{uf}(X_{uf} - w_{sf}) + f_{sf} = 0 \quad (3)$$

$$m_{ur} \ddot{X}_{ur} - c_{sr}(\dot{X}_{sr} - \dot{X}_{ur}) - k_{sr}(X_{sr} - X_{ur}) + k_{ur}(X_{ur} - w_{sr}) + f_{sr} = 0 \quad (4)$$

(b) Constraints

$X_s = (L_f X_{sr} + L_r X_{sf}) / L$

$\Theta_s = (X_{sf} - X_{sr}) / L$

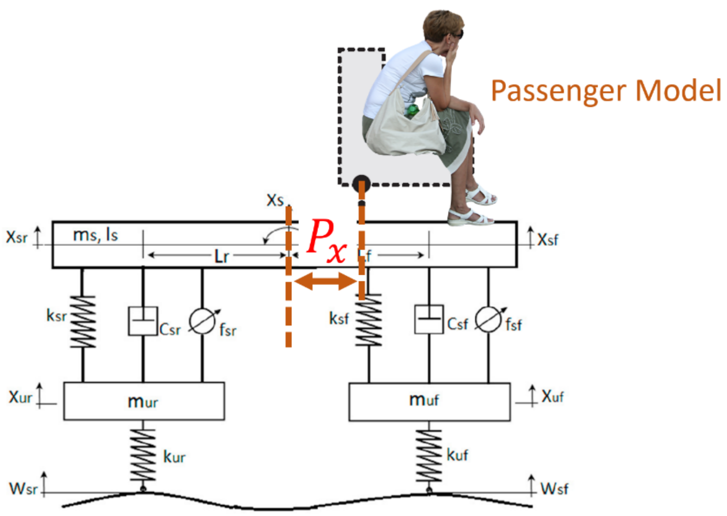
(c) State-space Representation

- $\dot{x} = Ax + Bu$
- $y = Cx + Du$

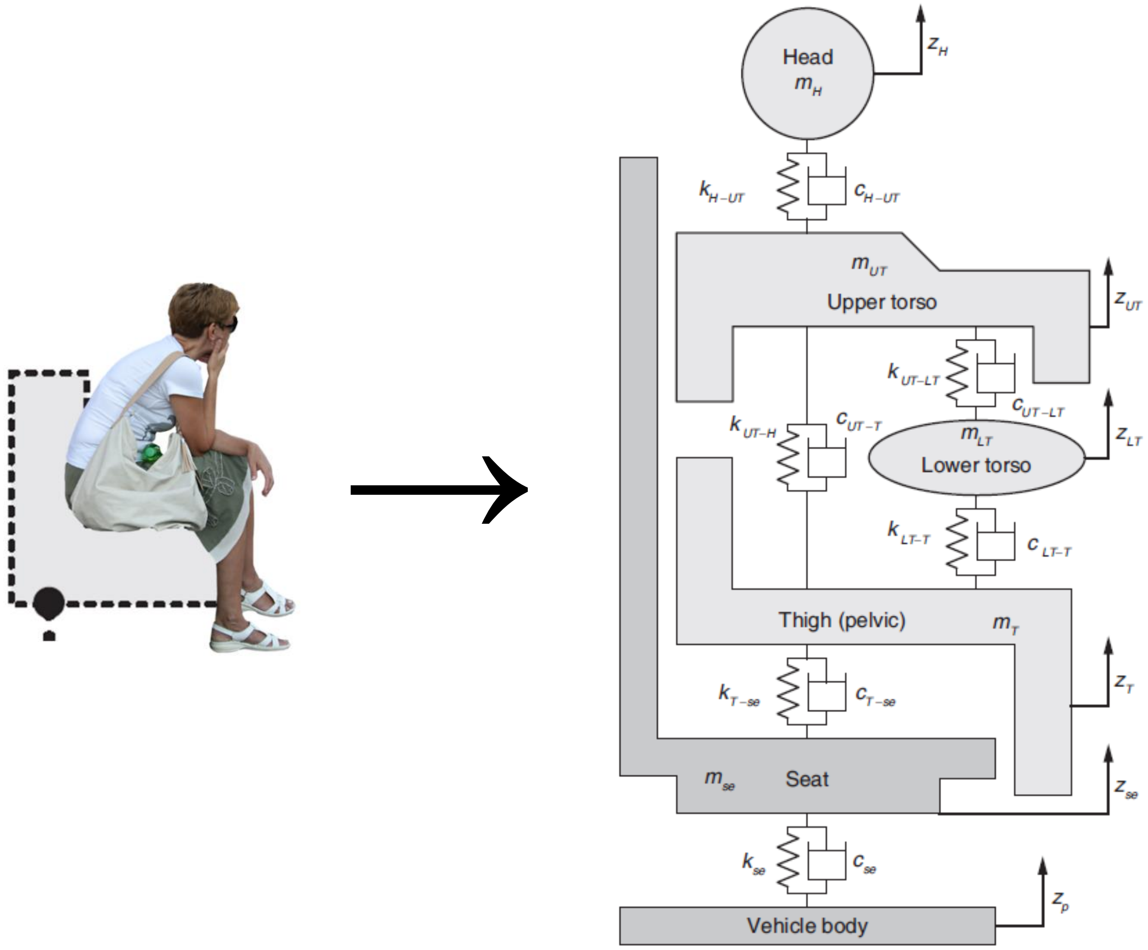
Where $x = [\dot{x_{sf}}\ \dot{x_{uf}}\ \dot{x_{sr}}\ \dot{x_{ur}}\ x_{sf}\ x_{uf}\ x_{sr}\ x_{ur}]^T$, and $u = [\ w_{sf}\ w_{sr}\ f_{sf}\ f_{sr}\ ]^T$

(2) Passenger Model

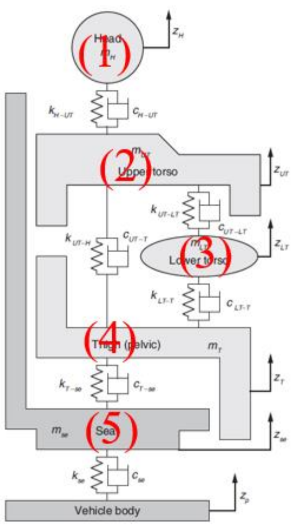
The passenger sitting in the car should also be modeled because there are uncertain parameters in the passenger model.



(a) Uncertain Biodynamics [2]



(b) Equations of Motion



$$m_H \ddot{z}_H = -k_{H-UT}\left(z_H - z_{UT}\right) - c_{H-UT}\left(\dot{z}_H - \dot{z}_{UT}\right) \quad (1)$$

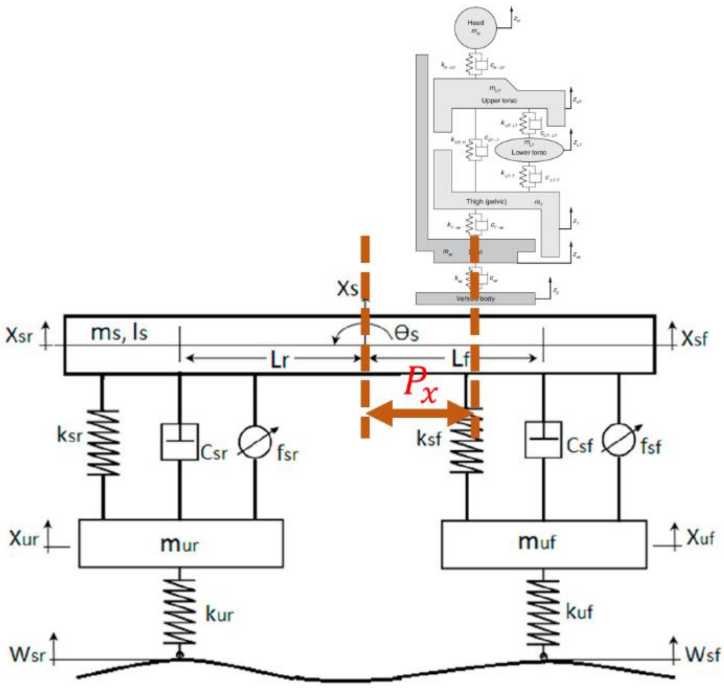
$$\begin{aligned} m_{UT}\ddot{z}_{UT} = {} & k_{H-UT}\left(z_H - z_{UT}\right) - k_{UT-LT}\left(z_{UT} - z_{LT}\right) - k_{UT-T}\left(z_{UT} - z_T\right) \\ & + c_{H-UT}\left(\dot{z}_H - \dot{z}_{UT}\right) - c_{UT-LT}\left(\dot{z}_{UT} - \dot{z}_{LT}\right) - c_{UT-T}\left(\dot{z}_{UT} - \dot{z}_T\right) \end{aligned} \quad (2)$$

$$\begin{aligned} m_{LT}\ddot{z}_{LT} = {} & k_{UT-LT}\left(z_{UT} - z_{LT}\right) - k_{LT-T}\left(z_{LT} - z_T\right) + c_{UT-LT}\left(\dot{z}_{UT} - \dot{z}_{LT}\right) \\ & - c_{LT-T}\left(\dot{z}_{LT} - \dot{z}_T\right) \end{aligned} \quad (3)$$

$$\begin{aligned} m_T\ddot{z}_T = {} & k_{UT-T}\left(z_{UT} - z_T\right) + k_{LT-T}\left(z_{LT} - z_T\right) - k_{T-se}\left(z_T - z_{se}\right) \\ & + c_{UT-T}\left(\dot{z}_{UT} - \dot{z}_T\right) + c_{LT-T}\left(\dot{z}_{LT} - \dot{z}_T\right) - c_{T-se}\left(\dot{z}_T - \dot{z}_{se}\right) \end{aligned} \quad (4)$$

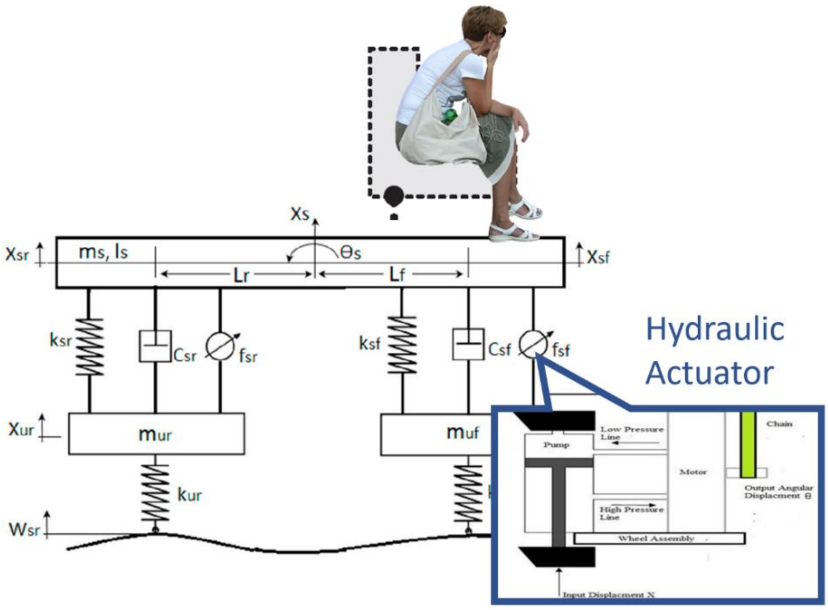
$$\begin{aligned} m_{se}\ddot{z}_{se} = {} & k_{T-se}\left(z_T - z_{se}\right) - k_{se}\left(z_{se} - z_p\right) + c_{T-se}\left(\dot{z}_T - \dot{z}_{se}\right) \\ & - c_{se}\left(\dot{z}_{se} - \dot{z}_p\right) \end{aligned} \quad (5)$$

(c) Relationship between the passenger and the half car



$$z_p = x_s + P_x\theta_s \quad (6)$$

(3) Hydraulic Actuators



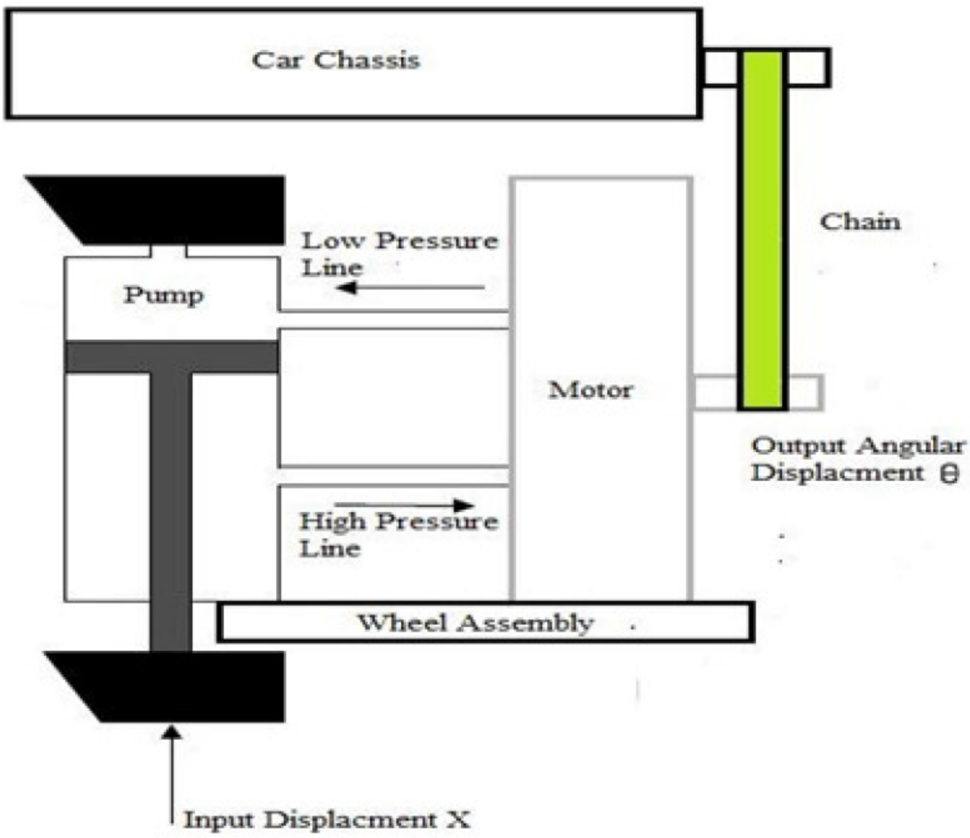


Figure 6: Hydraulic Actuator Block Diagram [3]

Oil waft from the pump, $q_p = K_p \frac{dx}{dt}$

Oil glide via the motor, $q_m = K_m \frac{d\theta}{dt}$

Leakage flow rate, $q_i = K_i P$

Compressibility flow rate, $q_c = K_c \frac{dP}{dt}$

The rate at which the oil flows from the pump is given by using the sum of the oil go with the flow fee through the motor, the leakage fee and the compressibility flow rate:

$$q_p = q_m + q_i + q_c$$

Then, we have

$$K_p \frac{dx}{dt} = K_m \frac{d\theta}{dt} + K_i P + K_c \frac{dP}{dt}$$

Jibril and other authors [3] simplify the equations as follows.

- Km = Kt = Kc
- Tm = Tl
- Kc = 0

Therefore, the transfer function is

$$\frac{\theta(s)}{X(s)} = \frac{K_p}{\left[\frac{K_i J}{K_m} s + \frac{K_m^2 + K_i B}{K_m}\right]}$$

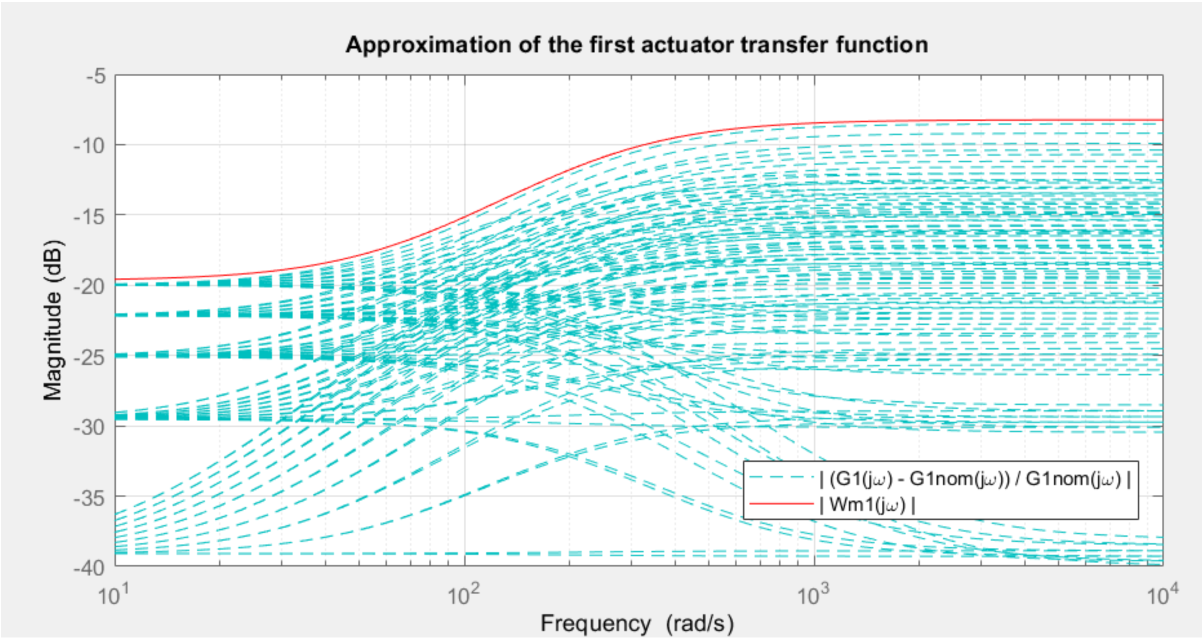
5. Uncertainty Model

(1) Uncertainties in Hydraulic Actuators

(a) nominal $G_1(s) = G_2(s) = \frac{1.08}{0.005s+1}$

(b) purturbed $\widetilde{G_i(s)} = \frac{K_i}{T_i s+1}$ with multiplicative uncertainty, i = 1, 2.

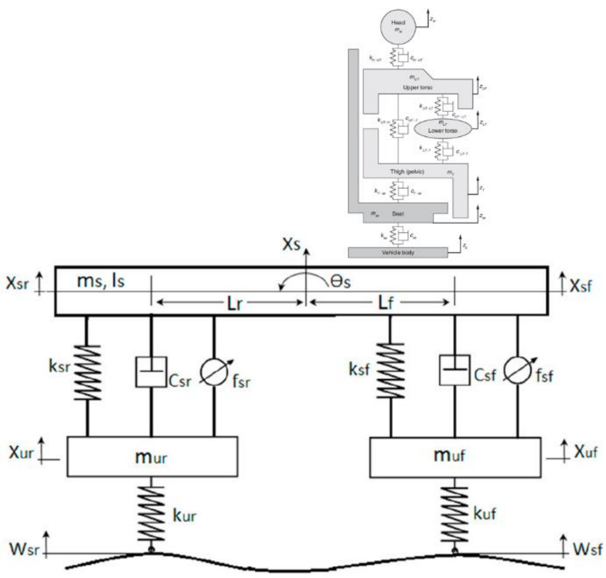
- $\tilde{G}_i = G_i\,(1 + W_{mi}\Delta_i)$, where $\|\Delta_i\|_\infty < 1$.
- K_i : uncertainty of 10%
- T_i : uncertainty of 20%



Multiplicative uncertainty: $\widetilde{G_i(s)} = G_i(s)(1 + W_{mi}\Delta_i)$,

where $W_{mi} = \frac{0.3803s + 60.8973}{s+599.5829}$, i = 1, 2.

(2) Uncertainties in The Half Car



The uncertainties in the system include all dampers in the passenger model and all parameters in two hydraulic actuators.

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Uncertain continuous-time state-space model with 3 outputs, 4 inputs, 22 states.
The model uncertainty consists of the following blocks:
  C_H_UT: Uncertain real, nominal = 310, variability = [-15,15]%, 1 occurrences
  C_LT_T: Uncertain real, nominal = 330, variability = [-15,15]%, 1 occurrences
  C_T_se: Uncertain real, nominal = 2.48e+03, variability = [-15,15]%, 1 occurrences
  C_UT_LT: Uncertain real, nominal = 200, variability = [-15,15]%, 1 occurrences
  C_UT_T: Uncertain real, nominal = 909, variability = [-15,15]%, 1 occurrences
  C_se: Uncertain real, nominal = 150, variability = [-15,15]%, 1 occurrences
  Delta_act1: Uncertain 1x1 LTI, peak gain = 1, 1 occurrences
  Delta_act2: Uncertain 1x1 LTI, peak gain = 1, 1 occurrences
  K1: Uncertain real, nominal = 1.08, variability = [-10,10]%, 1 occurrences
  K2: Uncertain real, nominal = 1.08, variability = [-10,10]%, 1 occurrences
  T1: Uncertain real, nominal = 0.005, variability = [-20,20]%, 1 occurrences
  T2: Uncertain real, nominal = 0.005, variability = [-20,20]%, 1 occurrences
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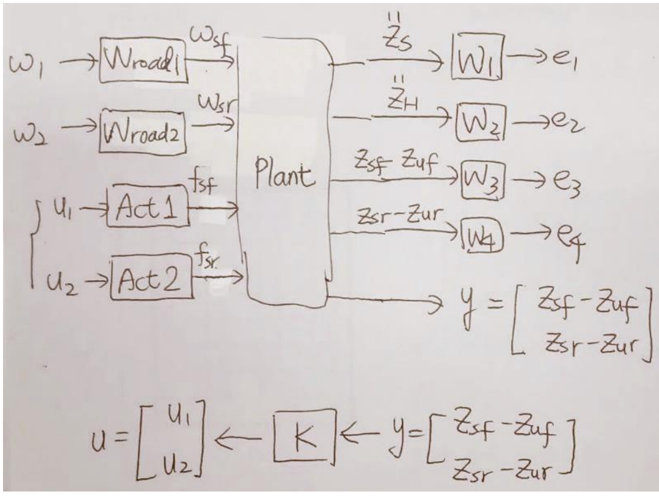
6. Robust Control Design

Recall that the robust performance can be achieved if the structured singular value is smaller than one.

(1) Block Diagram



(2) Plant P

Plant P = A Half-Car Active Suspension System

= Half-Car Model + Passenger Model + Hydraulic Actuator

State-Space representation of P:

$$\dot{x} = Ax + Bu$$

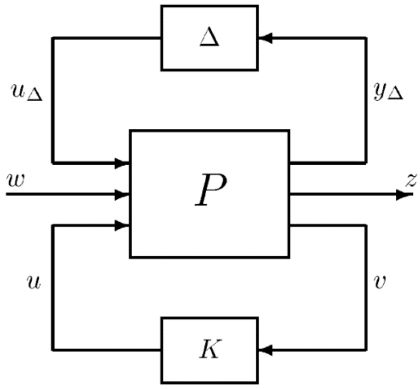
$$y = Cx + Du$$

Where $u^T = [\, w_{sf},\ w_{sr}\ ,\ f_{sf}\ ,\ f_{sr}\ ]$

$y^T = [\,(x_{sf} - x_{uf})\ ,\ (x_{sr} - x_{ur}),\ \ddot{z_H}]$

and $x^T = [\dot{x_{sf}}\ \dot{x_{uf}}\ \dot{x_{sr}}\ \dot{x_{ur}}\ x_{sf}\ x_{uf}\ x_{sr}\ x_{ur}\ Z_H\ Z_{UT}\ Z_{LT}\ Z_T\ Z_{se}\ \dot{Z_H}\ \dot{Z_{UT}}\ \dot{Z_{LT}}\ \dot{Z_T}\ \dot{Z_{se}}]$

(3) Linear Fractional Transformation (LFT)



- Performance output: $z^T = [\ddot{x}_s\,, \ddot{Z}_H\,, (x_{sf} - x_{uf})\ ,\ (x_{sr} - x_{ur})\ ]$
- The input of K: $v^T = [(x_{sf} - x_{uf})\ , (x_{sr} - x_{ur})]$

(4) Weighting Selection

$W_1(s) = \frac{0.72s}{s^2 + 7.2s + 5200}$ (band-pass filter)

(5) μ Synthesis

D-K ITERATION SUMMARY:

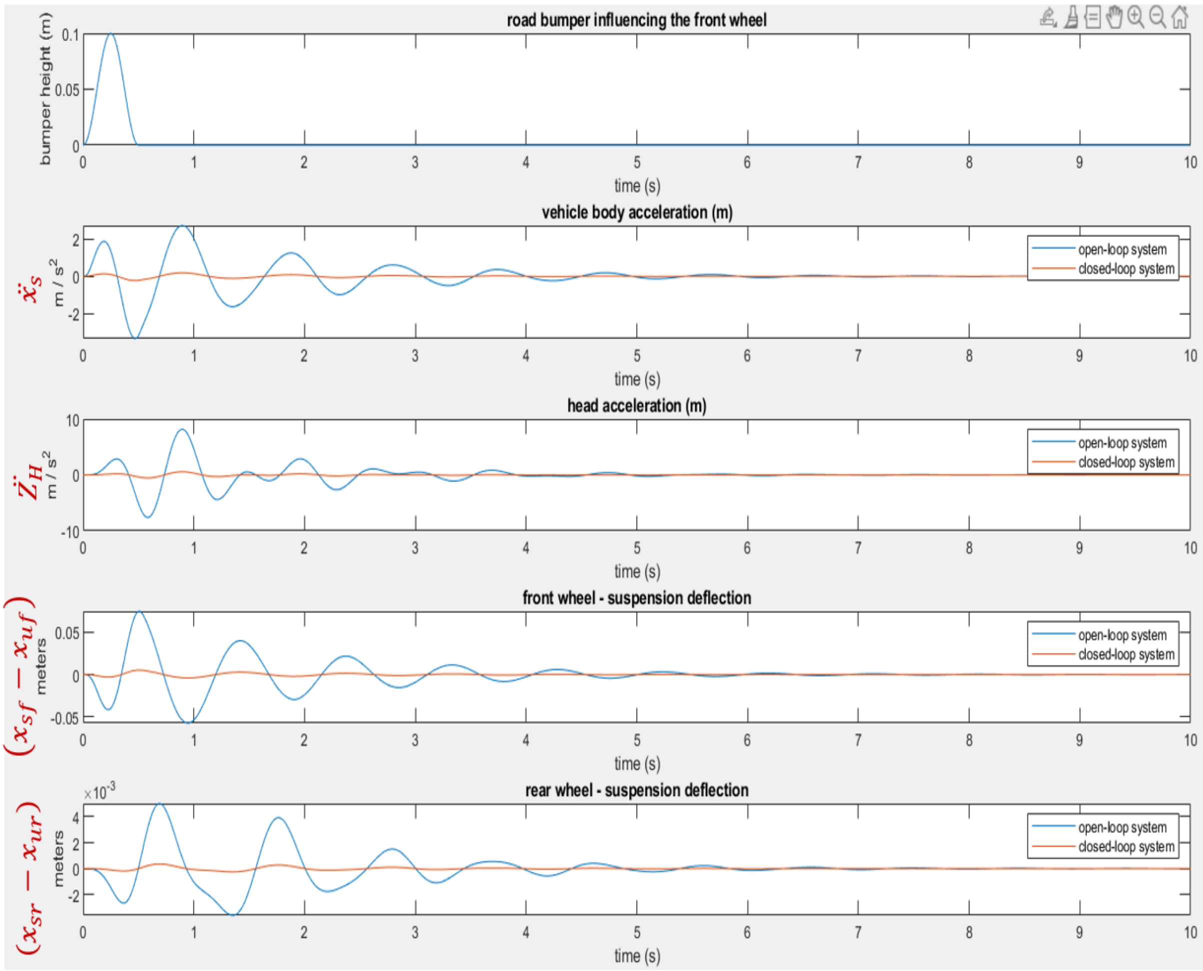
	Robust performance			Fit order
Iter	K Step	Peak MU	D Fit	D
1	2.46	0.301	0.3036	66
2	0.3036	0.301	0.3026	62
3	0.3026	0.301	0.3024	62

Best achieved robust performance: 0.301

The structured singular value μ = 0.301 < 1.

Therefore, the robust performance is achieved in my robust control design.

(6) Results



7. Conclusion

(1) This project establishes a robust feedback control synthesis for a class of half-car suspension systems considering a 4-DOF passenger's biodynamics with parametric uncertainties.

(2) The robust performance can be achieved through the design of μ synthesis, and both ride comfort and suspension deflections of the car are improved.

8. References

[1] Gandhi, Puneet & Sasidharan, Adarsh & Ramachandran, K.I.. (2017). Performance Analysis of Half Car Suspension Model with 4 DOF using PID, LQR, FUZZY and ANFIS Controllers. Procedia Computer Science. 115. 2-13. 10.1016/j.procs.2017.09.070.

[2] Gudarzi M, Oveisi A. Robust Control for Ride Comfort Improvement of an Active Suspension System considering Uncertain Driver's Biodynamics. Journal of Low Frequency Noise, Vibration and Active Control. 2014;33(3):317-339. doi:10.1260/0263-0923.33.3.317

[3] Jibril, Mustefa (2020). H ∞ and μ-synthesis Design of Quarter Car Active Suspension System. International Journal of Scientific Research and Engineering Development 3 (1):608-619. PhilArchive copy v1: https://philarchive.org/archive/JIBHAv1

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